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Global minimizers for free energies of subcritical aggregation equations with degenerate diffusion

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ABSTRACT

We prove the existence of global minimizers of a class of free energies related to aggregation equations with degenerate diffusion on $\mathbb{R}^d$. Such equations arise in mathematical biology as models for organism group dynamics which account for competition between the tendency to aggregate into groups and nonlinear diffusion to avoid overcrowding. The existence of non-trivial stationary solutions with minimal energy representing coherent groups in $\mathbb{R}^d$ is therefore of interest. A scaling criticality that measures the balance between the diffusive and aggregative forces as mass spreads is shown to govern the existence and non-existence of global minimizers. The primary difficulty confronted here is the inability to verify strict subadditivity conditions for biologically relevant problems which violate homogeneity-type assumptions known to be sufficient. To recover, we show that sufficiently degenerate diffusion provides a weaker condition from which tightness of symmetrized infimizing sequences can be recovered, even when the nonlocal attractive force is extremely weak.

1. Introduction

We study the global minimizers of the non-convex 'free energy'

$$\mathcal{F}(u) = \int_{\mathbb{R}^d} \Phi(u)\mathrm{d}x - \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} u(x)u(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y := S(u) - \frac{1}{2}\mathcal{W}(u), \tag{1}$$

with $u \in L^1_+(\mathbb{R}^d) := \left\{u \in L^1(\mathbb{R}^d) : u \geq 0\right\}$ subject to the constraint $\|u\|_1 := M$ over $\mathbb{R}^d$ with $d \geq 2$. We are interested in determining the choices of $\mathcal{K}$, Φ and M for which there exist global minimizers with mass M. We refer to $S(u)$ as the entropy and $\mathcal{W}(u)$ as the interaction energy. Energies such as (1) arise in the study of aggregation equations and Patlak–Keller–Segel models with degenerate diffusion [1–10]. A typical example is

$$u_t + \nabla\cdot(u\nabla\mathcal{K} * u) = \Delta u^m \tag{2}$$

for $m > 1$ and $u(t) \in L^1_+(\mathbb{R}^d)$. These equations are formally a gradient flow for (1) with $\Phi(u) = \frac{1}{m-1}u^m$ under the Euclidean Wasserstein distance [11–13]; however, as in many applications (1) is not displacement convex in the sense of [14], the established theory does not directly apply. Numerical simulations indicate that for certain choices of $\mathcal{K}$ and m, there are compactly supported stationary solutions to (2) that are local attractors [15,2]. The purpose of this work is to study the existence of global minimizers of (1) which are, in particular, minimal energy stationary solutions of PDEs such as (2).

Energy functionals of the general form (1) have been studied in different contexts, as these or similar problems arise in the Chandrasekhar theory of celestial mechanics [16–19] and the McKean–Vlasov equation in statistical mechanics (see [20]

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and the references therein). The existence and non-existence of minimizers in critical and subcritical problems for the 3D case with $\mathcal{K} = |x|^{-1}$ has been investigated by Lions, by Lieb and Yau, and by Blanchet et al. using symmetrization techniques [17,19,4] (see also [16] for a limiting method applied to related rotating star models). The problem with more general $\mathcal{K}$ on $\mathbb{R}^d$ has been confronted by Lions via symmetrization methods in [17], and in [18] using concentration compactness, the latter requiring no monotonicity, symmetry or regularity assumptions on $\mathcal{K}$. Applying the symmetric decreasing rearrangement [21] provides additional compactness; however, tightness in L^1 of minimizing sequences is not obtained directly and additional homogeneity-type assumptions are sometimes necessary to ensure that minimizers satisfy the mass constraint [17]. Concentration compactness provides a more direct and intuitive way of evaluating the tightness of infimizing sequences via the necessary and sufficient conditions of strict subadditivity. However, as discussed in [22], verifying these conditions is not generally straightforward, and often requires homogeneity-type assumptions (as is the case in [18, Corollary II.1]).

In this work we focus on the cases which violate the homogeneity assumptions made in [17,18]. These cases do not seem to appear in the literature for two reasons. First, while they are of interest for biological applications, they fall outside the majority of physical applications areas, where the homogeneity assumptions generally hold. Indeed, cases of most biological interest often have high-order degenerate diffusion and kernels which decay quickly in space (see [15,2,5] and the references therein). Second, more refined techniques are required to treat them, as strict subadditivity does not seem to follow from simple arguments. In our work, a scaling argument discussed below identifies the regimes in which minimizers satisfying the mass constraint are expected to exist. The tightness of symmetrized sequences is verified with a combination of concentration compactness and the additional control provided by symmetry. This will allow us to construct global minimizers even for weak, short-range nonlocal attractive forces provided Φ vanishes sufficiently fast at zero. However, in the absence of strict subadditivity, it is possible that non-symmetrized infimizing sequences may still fail to be precompact in L^1. Hence, the assumption that $\mathcal{K}$ is radially symmetric and nonincreasing (see (3)), which implies that the nonlocal interaction is purely attractive, is essential for our work and we are not aware of whether similar results hold more generally. The minimizer problem without diffusion and with a more general attractive–repulsive $\mathcal{K}$ has been studied elsewhere (see e.g. [23,24]).

The kernels that we consider in this work satisfy the properties

$$\mathcal{K}(x) = \mathcal{K}(|x|) \geq 0, \quad \mathcal{K}(|x|) \text{ nonincreasing}; \tag{3}$$

$$\mathcal{K}(x) \in \mathcal{K} \in L^{p,\infty}(B_1(0)) \cap L^{\hat{p}}(\mathbb{R}^d \setminus B_1(0)), \quad 1 < p \leq \infty, 1 \leq \hat{p} < \infty. \tag{4}$$

Here, p roughly measures how singular $\mathcal{K}$ is and $\hat{p}$ measures how quickly $\mathcal{K}$ decays at infinity. In what follows we define the critical exponent $m^\star = (p+1)/p$. Recall the following well known inequality which is a generalization of the Hardy–Littlewood–Sobolev inequality: for all $f \in L^1 \cap L^{m^\star}$ and $\delta > 0$,

$$\iint f(x)f(y)\mathcal{K}(x-y)\mathbf{1}_{B_\delta}(|x-y|)\mathrm{d}x\mathrm{d}y \leq C_0 \|\mathcal{K}\mathbf{1}_{B_\delta}\|_{L^{p,\infty}} \|f\|_1^{2-m^\star} \|f\|_{m^\star}^{m^\star}, \tag{5}$$

where C_0 is a constant that depends on p and d. Similar inequalities are used in [4,18,1]. Define

$$\mathcal{Y}_M := \left\{ u \in L^1_+ \cap L^{m^\star} : \|u\|_1 = M \right\} \quad \text{and} \quad I_M := \inf_{u \in \mathcal{Y}_M} \mathcal{F}(u).$$

The energies that we consider are subcritical in the sense that they satisfy

$$\liminf_{z \to \infty} \frac{\Phi(z)}{z^{m^\star}} > \frac{1}{2} C_0 \|\mathcal{K}\mathbf{1}_{B_\delta}\|_{L^{p,\infty}} M^{2-m^\star}. \tag{6}$$

Note that the quantity on the left hand side need not be finite. The purpose of this assumption, also made in [18], is to ensure that $-\infty < I_M$ and that sequences $\{u_n\} \subset \mathcal{Y}_M$ with $\sup \mathcal{F}(u_n) < \infty$ have uniformly bounded entropy and $L^{m^\star}$ norms (see for instance [18] or [1]). We point out that this condition may not be sharp in general, but the precise constant here is not the emphasis of this work; see [4,1]. Moreover, we assume that

$$\Phi(z) : [0, \infty) \to \mathbb{R}, \quad \text{non-negative, strictly convex} \tag{7}$$

$$\Phi(z) = \chi z^2 + o(z^2), \quad z \to 0, \text{ for some } \chi, 0 \leq \chi < \infty. \tag{8}$$

Note that these conditions imply that $\Phi(z)$ is strictly increasing and $\lim_{z \to 0} \Phi(z) z^{-1} = 0$.

Theorem 1. *Suppose that $M > 0$ and that $\mathcal{K}$ and Φ satisfy conditions (3), (4) and (6)–(8). Suppose further that one of the following holds:*

(i) $\chi = 0$;
(ii) $0 < \chi < \infty$ *and* $2\chi < \|\mathcal{K}\|_1 \leq \infty$.

Then $I_M < 0$ and there exists a radially symmetric and nonincreasing $u^\star \in \mathcal{Y}_M$ such that $\mathcal{F}(u^\star) = I_M$. If $\mathcal{K}$ is strictly radially decreasing, then all global minimizers are radially symmetric and nonincreasing.

Remark 1. As far as the author is aware, due to the lack of convexity, uniqueness is largely unresolved except when $\mathcal{K}$ is the Newtonian potential [19] or has similar monotonicity properties [25]. In critical cases, minimizers are not unique at the critical mass [17,4].

Remark 2. In applications, Φ is often negative near zero [26]. For problems with degenerate diffusion, one can show that $S(u) \gtrsim -M$, and the methods here apply simply on modifying (1) by including a constant depending on M. However, for problems with more general Φ, such as the Boltzmann entropy $\Phi(u) = u \log u$, the methods here would have to be modified. Similarly, modifications have to be made to treat potentials $\mathcal{K}$ which are unbounded from below, such as the logarithmic potential. The primary difficulty in these cases is the verification that $I_M < 0$, as in these cases, the scaling argument used to verify this fact clearly breaks down.

Before beginning the proof of Theorem 1, we describe the scaling criticality associated with (1) which dictates the main existence results in this work. A similar scaling heuristic also appears in [18], Remark II.4, and our work in some sense represents the end-point of that analysis carried to the inhomogeneous limit. Moreover, our analysis reveals a kind of critical threshold different than those analyzed in [17,18]. The results in [17] and the profile decomposition in [18] applied to sequences in $\mathcal{Y}_M$ with bounded $L^{m^\star}$ norm suggest that the primary difficulty for proving that I_M is attained for some $u^\star \in \mathcal{Y}_M$ will be ensuring that the mass of infinimizing sequences does not split apart or vanish. Naturally, this leads to considering the relative balance of the entropy and interaction energy as the mass of a sequence in $\mathcal{Y}_M$ spreads out. For simplicity, consider the case $S(u) = \frac{1}{m-1}\int u^m \mathrm{d}x$ with $m > m^\star \geq 1$, and suppose that $\mathcal{K} \in L^1$. Suppose that $u(x) \in L^1_+ \cap L^m$ and define $u_\lambda(x) = \lambda^d u(\lambda x)$. Then,

$$\mathcal{F}(u_\lambda) = \frac{\lambda^{dm-d}}{m-1}\|u\|_m^m - \frac{\lambda^d}{2}\iint u(x)u(y)\frac{1}{\lambda^d}\mathcal{K}\left(\frac{x-y}{\lambda}\right)\mathrm{d}x\mathrm{d}y.$$

The key observation is that as $\lambda \to 0$, the second term behaves like $(\lambda^d/2)\|\mathcal{K}\|_1\|u\|_2^2$. If $m > 2$ then for sufficiently small λ, $\mathcal{F}(u_\lambda) < 0$, whereas if $m < 2$, this is no longer true. However, the $m < 2$ case can be recovered by requiring sufficiently large mass [17,18]. The case $m = 2$ is in some sense critical, since the entropy and interaction energy scale the same as $\lambda \to 0$, and the sign in the limit only depends on the value of $\|\mathcal{K}\|_1$. Indeed, we shall see that in this case, the existence of minimizers is dictated only by $\|\mathcal{K}\|_1$, a critical threshold which to the author's knowledge does not appear elsewhere in the literature. The corresponding scaling argument in Remark II.4 of [18] for kernels with certain homogeneity assumptions reveals a more standard 'critical mass' condition which requires the mass of potential minimizers to be sufficiently large. Note also that the above scaling argument shows that if $\lim_{z\to 0}\Phi(z)z^{-1} = 0$ then $I_M \leq 0$ for all $M \geq 0$ as $\lim_{\lambda\to 0}\mathcal{F}(u_\lambda) = 0$. The case $m > 2$ is important for biological applications, and is also the case in which equation (2) may be formally rewritten as a regularized nonlocal interface problem [5].

The condition $\|\mathcal{K}\|_1 > 2\chi$ in (ii) is close to sharp. In general, minimizers may not exist for any value of $M > 0$ if this is violated due to the possibility of every non-zero function having strictly positive energy.

Proposition 1. *Suppose that $\Phi(z) \geq z^2$ and $\|\mathcal{K}\|_1 < 2$. Then for all $M > 0$, $I_M = 0$ and there exist no non-zero global minimizers of the free energy.*

Proof. Recall from above that any global minimizer $u^\star \in \mathcal{Y}_M$ will satisfy $\|u^\star\|_2^2 \leq S(u^\star) < \infty$. By the Cauchy–Schwarz inequality and Young's inequality for convolutions,

$$\mathcal{F}(u^\star) \geq \|u^\star\|_2^2 - \frac{1}{2}\iint u^\star(x)u^\star(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y \geq \left(1 - \frac{\|\mathcal{K}\|_1}{2}\right)\|u^\star\|_2^2.$$

This is clearly a contradiction unless $\|u^\star\|_2^2 = 0$, as $I_M \leq 0$ for all $M \geq 0$. $\square$

Moving towards the proof of the main theorem, we state and prove two lemmas, beginning with a refinement of the scaling analysis discussed above.

Lemma 1 (*Scaling Lemma*)**.** *Let* (i) *or* (ii) *hold. Then* $\forall M > 0, \exists \phi \in C_c^\infty \cap \mathcal{Y}_M$ *with* $\mathcal{F}(\phi) < 0$.

Proof. Suppose that $\phi \in C_c^\infty \cap \mathcal{Y}_M$ and consider the mass-invariant scaling $\phi_\lambda(x) = \lambda^d\phi(\lambda x)$. Then for $R > 0$,

$$\begin{aligned}
\mathcal{F}(\phi_\lambda) &= S(\phi_\lambda) - \frac{\lambda^{2d}}{2}\iint \phi(\lambda x)\phi(\lambda y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y \\
&= \lambda^d\left(\int \frac{\Phi(\lambda^d\phi(\lambda x))}{\lambda^d}\mathrm{d}x - \frac{1}{2}\iint \phi(x)\phi(y)\frac{1}{\lambda^d}\mathcal{K}\left(\frac{x-y}{\lambda}\right)\mathrm{d}x\mathrm{d}y\right) \\
&\leq \lambda^d\left(\int \frac{\Phi(\lambda^d\phi(\lambda x))}{\lambda^d}\mathrm{d}x - \frac{1}{2}\iint \phi(x)\phi(y)\frac{1}{\lambda^d}\mathcal{K}\left(\frac{x-y}{\lambda}\right)\mathbf{1}_{B_R}(|x-y|)\,\mathrm{d}x\mathrm{d}y\right).
\end{aligned}$$

By (8) and $\phi \in C_c^\infty$, for all $\epsilon > 0$ and λ sufficiently small (depending on ϕ) such that

$$\int \lambda^{-d}\Phi(\lambda^d\phi(\lambda x))\mathrm{d}x = \int \lambda^{-2d}\Phi(\lambda^d\phi(x))\mathrm{d}x \le \chi\|\phi\|_2^2 + \epsilon\|\phi\|_2^2,$$

Therefore, since $\mathcal{K} \in L^1_{\mathrm{loc}}$ and $\phi \in C_c^\infty$, for all ϵ we may pick λ sufficiently small such that

$$\mathcal{F}(\phi_\lambda) \le \lambda^d\left(\chi - \frac{\|\mathcal{K}\mathbf{1}_{B_{R/\lambda}}\|_1}{2}\right)\|\phi\|_2^2 + \epsilon\lambda^d.$$

Therefore, if $\chi = 0$, we clearly have $I_M < 0$. Moreover, if $0 < \chi < \infty$, then the sign of the right hand side does not depend on λ or ϕ, and is negative only if $\|\mathcal{K}\mathbf{1}_{B_{R/\lambda}}\|_1 > 2\chi$. By choosing R sufficiently large, we see that this is equivalent to $\|\mathcal{K}\|_1 > 2\chi$. □

Lemma 1 provides conditions under which $I_M < 0$, but in general this is insufficient to imply the existence of non-trivial minimizers. Without strict subadditivity, we cannot directly apply the results of [18] to prove Theorem 1. To recover, we will use a symmetrization argument and the following lemma, which in general is strictly weaker than subadditivity.

Lemma 2. *Let* (i) *or* (ii) *hold and* $M_1 > M_2$. *Then* $I_{M_1} < I_{M_2}$.

Proof. Let $u_n \in \mathcal{Y}_{M_2}$ be such that $\lim_{n\to\infty}\mathcal{F}(u_n) \to I_{M_2}$. By Lemma 1, there exists $v \in C_c^\infty \cap \mathcal{Y}_M$ such that $\mathcal{F}(v) < 0$ and $\|v\|_1 = M_1 - M_2$. Without loss of generality, by the Riesz symmetric decreasing rearrangement and as $\mathcal{K}$ is radially symmetric and nonincreasing, we may take u_n and v to be radially symmetric and nonincreasing, since applying a symmetric rearrangement will only decrease the interaction energy and leaves the entropy unchanged (Theorem 3.7 of [21]). Hence, since each u_n is radially symmetric and nonincreasing we have $u_n(x) \lesssim M_2|x|^d$; hence for all ϵ, $\exists R(\epsilon) > 0$ such that $|u_n(x)| < \epsilon$ for all $|x| > R$ (note that this implies tightness in L^q for all $1 < q \le \infty$ but not L^1). Choose x_n such that $B_R(x_n) \cap \mathrm{supp}\,v = \emptyset$ and suppose that $\hat{u}_n = u_n(\cdot - x_n)$. Define

$$z_n(x) = v(x) + \hat{u}_n(x). \tag{9}$$

By $v, \hat{u}_n \ge 0$ we have $\|z_n\|_1 = M_1$. Now by the approximately disjoint supports,

$$\begin{aligned}\mathcal{F}(z_n) &= \int \Phi(z_n(x))\mathrm{d}x - \frac{1}{2}\iint z_n(x)z_n(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y \\ &\le \int \Phi(u_n(x))\mathrm{d}x + \int_{\mathrm{supp}\,v}\Phi(v(x)+\epsilon)\mathrm{d}x - \frac{1}{2}\iint z_n(x)z_n(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Notice that since $v \in C_c^\infty$, by the mean value theorem and the convexity of Φ,

$$\int_{\mathrm{supp}\,v}\Phi(v(x)+\epsilon)\mathrm{d}x \le \int \Phi(v(x))\mathrm{d}x + \epsilon\Phi'(\|v\|_\infty)\,|\mathrm{supp}\,v|\,.$$

Therefore, by $\mathcal{K} \ge 0$,

$$\mathcal{F}(z_n) \le \mathcal{F}(u_n) + \mathcal{F}(v) + \epsilon\Phi'(\|v\|_\infty)\,|\mathrm{supp}\,v|\,.$$

Since $\mathcal{F}(v) < 0$ we may choose ϵ sufficiently small to ensure that $\liminf_{n\to\infty}\mathcal{F}(z_n) < \liminf_{n\to\infty}\mathcal{F}(u_n) = I_{M_2}$. □

Proof of Theorem 1. We now prove that conditions (i) and (ii) imply the stated results.

Let $u_n \in \mathcal{Y}_M$ be such that $\mathcal{F}(u_n) \to I_M$. Recall that by (6), the sequence u_n is uniformly bounded in $L^{m^\star}$. As above, by the symmetric decreasing rearrangement, we may assume that u_n is radially symmetric and nonincreasing. Moreover, the strict rearrangement inequality (Theorem 3.9 of [21]) proves that if $\mathcal{K}$ is radially symmetric and strictly decreasing then all minimizers are radially symmetric and nonincreasing. We now show that $\{u_n\}$ has a convergent subsequence in the strong L^1 topology.

We follow the approach of Theorem II.1 of [18]. Following the work contained therein, tightness up to translation is established using the profile decomposition lemma (Lemma I.1). Accordingly, there exists a subsequence of $\{u_n\}$, not relabeled, such that one of the following three possibilities occurs:

(i) Tight up to translation: $\exists\{y_n\} \subset \mathbb{R}^d$ for which $\forall\epsilon > 0$, $\exists R > 0$ such that $\int_{\mathbb{R}^d\setminus B_R(y_n)} u_n\mathrm{d}x < \epsilon$.

(ii) Vanishing: $\forall R > 0$, $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^d}\int_{B_R(y)} u_n\mathrm{d}x = 0$.

(iii) Dichotomy: $\forall\epsilon > 0$, $\exists\left\{u_n^{1,\epsilon}\right\}, \left\{u_n^{2,\epsilon}\right\}, \left\{v_n^\epsilon\right\} \subset L^1_+$, such that $u_n = u_n^{1,\epsilon} + u_n^{2,\epsilon} + v_n^\epsilon$ with, for $i \in \{1,2\}$, $u_n^{1,\epsilon}u_n^{2,\epsilon} = u_n^{i,\epsilon}v_n^\epsilon \equiv 0$, $u_n^{i,\epsilon}, v_n^\epsilon \le u_n$, $\lim_{n\to\infty}\mathrm{dist}(\mathrm{supp}\,u_n^{1,\epsilon}, \mathrm{supp}\,u_n^{2,\epsilon}) = \infty$ and $\left|\lim_{n\to\infty}\|u_n^{1,\epsilon}\|_1 - M - \alpha\right| < \epsilon$, $\left|\lim_{n\to\infty}\|u_n^{2,\epsilon}\|_1 - \alpha\right| < \epsilon$ and $\lim_{n\to\infty}\|v_n^\epsilon\|_1 < \epsilon$ for some α, $0 < \alpha < M$.

Hence, it suffices to rule out the latter two possibilities.

Vanishing does not occur:

Vanishing is ruled out by $I_M < 0$. Indeed, $I_M < 0$ implies $\lim_{n\to\infty} \mathcal{W}(u_n) > 0$. Assume for a contradiction that the subsequence (not relabeled) $\{u_n\}$ vanishes as $n \to \infty$. Suppose that $q \in [(p+1)/2, p)$, $R > 1$; by Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
\mathcal{W}(u_n) &= \iint u_n(x)u_n(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y \\
&\le \|u_n\|^2_{2q/(2q-1)}\|\mathcal{K}\mathbf{1}_{B_{R^{-1}}}\|_q + \iint_{R^{-1}<|x-y|\le R} u_n(x)u_n(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y + M^2\|\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_R}\|_\infty \\
&\le \|u_n\|^2_{2q/(2q-1)}\|\mathcal{K}\mathbf{1}_{B_{R^{-1}}}\|_q + \|\mathcal{K}\mathbf{1}_{B_R\setminus B_{R^{-1}}}\|_\infty \int u_n(x)\int_{|x-y|<R} u_n(y)\mathrm{d}y\mathrm{d}x + M^2\|\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_R}\|_\infty \\
&\le \|u_n\|^2_{2q/(2q-1)}\|\mathcal{K}\mathbf{1}_{B_{R^{-1}}}\|_q + \|\mathcal{K}\mathbf{1}_{B_R\setminus B_{R^{-1}}}\|_\infty M \sup_{x\in\mathbb{R}^d}\left(\int_{|x-y|<R} u(y)\mathrm{d}y\right) + M^2\|\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_R}\|_\infty.
\end{aligned}$$

By interpolation, $\|u_n\|_{2q/(2q-1)}$ is uniformly bounded, since $2q/(2q-1) \in (1, m^\star]$ and u_n is uniformly bounded in $\mathcal{Y}_M$. Since $\{u_n\}$ vanishes we may deduce that

$$\liminf_{n\to\infty} \mathcal{W}(u_n) \lesssim \|\mathcal{K}\mathbf{1}_{B_{R^{-1}}}\|_q + \|\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_R}\|_\infty.$$

As $R \to \infty$, the last term vanishes since $\mathcal{K} \in L^{\hat{p}}(\mathbb{R}^d \setminus B_1(0))$ and is radially symmetric and nonincreasing and the first vanishes by the dominated convergence theorem and $\mathcal{K} \in L^q_{\mathrm{loc}}$. Therefore, we have deduced

$$\liminf_{n\to\infty} \mathcal{W}(u_n) \le 0,$$

which is a contradiction to $I_M < 0$.

Dichotomy does not occur:

Although we do not have strict subadditivity, we will take advantage of the weaker property, Lemma 2, along with radial symmetry, to rule out dichotomy. Suppose for a contradiction that dichotomy occurs. The properties of the decomposition $u_n = u_n^{1,\epsilon} + u_n^{2,\epsilon} + v_n^\epsilon$ together with u_n radially symmetric and nonincreasing (and therefore $u_n(x) \lesssim M|x|^{-d}$), one of $u_n^{1,\epsilon}$ or $u_n^{2,\epsilon}$ converges to zero in L^∞. In particular, one of the sequences must vanish, which is the key advantage of radial symmetry and the only place in this work where it is really necessary. Assume without loss of generality that $u_n^{2,\epsilon} \to 0$. For the remainder of the proof we will drop the explicit dependence on ϵ for the sake of notational brevity. By the disjoint supports,

$$S(u_n) \ge S(u_n^1) + S(u_n^2). \tag{10}$$

Moreover,

$$\begin{aligned}
\mathcal{W}(u_n) = \;& \mathcal{W}(u_n^1) + \mathcal{W}(u_n^2) + \mathcal{W}(v_n) + \iint u_n^1(x)u_n^2(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y \\
&+ \iint v_n(x)u_n^1(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y + \iint v_n(x)u_n^2(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

By (5) and interpolation, for any $\delta > 0$,

$$\begin{aligned}
\mathcal{W}(u_n^2) &\lesssim \|\mathcal{K}\mathbf{1}_{B_\delta}\|_{L^{p,\infty}}\|u_n^2\|_1^{2-m^\star}\|u_n^2\|_{m^\star}^{m^\star} + \|\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_\delta}\|_{\hat{p}}\|u_n^2\|^2_{\frac{2\hat{p}}{2\hat{p}-1}} \\
&\lesssim \|\mathcal{K}\mathbf{1}_{B_\delta}\|_{L^{p,\infty}}\|u_n^2\|_1^{3-m^\star}\|u_n^2\|_\infty^{m^\star-1} + \|\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_\delta}\|_{\hat{p}}\|u_n^2\|_1^{2-\frac{1}{\hat{p}}}\|u_n^2\|_\infty^{\frac{1}{\hat{p}}}.
\end{aligned} \tag{11}$$

Similarly for any $\delta > 0$,

$$\mathcal{W}(v_n) \le \|v_n\|_1^2\|\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_\delta}\|_\infty + \|\mathcal{K}\mathbf{1}_{B_\delta}\|_{L^{p,\infty}}\|v_n\|_1^{2-m^\star}\|v_n\|_{m^\star}^{m^\star}, \tag{12}$$

and for $i \in \{1, 2\}$,

$$\begin{aligned}
&\iint v_n(x)u_n^i(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y \\
&\le \|v_n\|_1\|u_n^i\|_1\|\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_\delta}\|_\infty + \|\mathcal{K}\mathbf{1}_{B_\delta}\|_{L^{p,\infty}}\|v_n\|_1^{1-m^\star/2}\|v_n\|_{m^\star}^{m^\star/2}\|u_n^i\|_1^{1-m^\star/2}\|u_n^i\|_{m^\star}^{m^\star/2}.
\end{aligned} \tag{13}$$

Finally, suppose that $d_n = \mathrm{dist}(\mathrm{supp}\, u_n^1, \mathrm{supp}\, u_n^2)$. Therefore,

$$\iint u_n^1(x)u_n^2(y)\mathcal{K}(x-y)\mathrm{d}x\mathrm{d}y = \iint u_n^1(x)u_n^2(y)\mathcal{K}(x-y)\mathbf{1}_{\mathbb{R}^d\setminus B_{d_n}}(|x-y|)\mathrm{d}x\mathrm{d}y \leq M^2\|\mathcal{K}\mathbf{1}_{\mathbb{R}^d\setminus B_{d_n}}\|_\infty. \tag{14}$$

Putting the estimates (10) and (14) together with $d_n \to \infty$, $\mathcal{K}$ radially symmetric and nonincreasing and $\mathcal{K} \in L^{\hat{p}}(\mathbb{R}^d\setminus B_1(0))$, along with $\lim_{n\to\infty}\|u_n^2\|_\infty = \lim_{n\to\infty} S(u_n^2) = 0$, $\lim_{n\to\infty}\|v_n\|_1 < \epsilon$ and the uniform boundedness of $\|u_n^i\|_{m^\star}$, $\|v_n\|_{m^\star}$, we infer that

$$I_M = \lim_{n\to\infty}\mathcal{F}(u_n) \geq \liminf_{n\to\infty}\mathcal{F}(u_n^1) - C\epsilon^{1-m^\star/2} \geq I_{M-\alpha} - C\epsilon^{1-m^\star/2},$$

for some C independent of ϵ. Taking $\epsilon \to 0$ contradicts Lemma 2 and rules out dichotomy, which implies that there is a subsequence of $\{u_n\}$ which is tight up to translation.

Conclusion of the proof:

Following [18] one may now prove without modification that tightness is sufficient to extract a subsequence (not relabeled) such that $u_n \to u^\star$ strongly in L^1 for some $u^\star \in \mathcal{Y}_M$ with $\mathcal{F}(u^\star) = I_M$. Strong convergence can be deduced from $\mathcal{F}(u_n) \to I_M$ and the strict convexity of $\Phi(u)$ (see [18]). This concludes the proof that assumptions (i) and (ii) imply the stated results. □

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